

DISCOVERING INTEGRALS WITH GEOMETRY

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ABSTRACT: In this paper, students and teachers are provided with problems that lead them in finding the integrals of logarithmic and inverse trigonometric functions early in the calculus sequence by using the Fundamental Theorem of Calculus and the concept of area and without the use of integration by parts. The methods link geometric and symbolic representations, and allow students to visually interpret these concepts.

KEYWORDS: Integration, logarithms, inverse trigonometric functions, Fundamental Theorem of Calculus.

For the past decade, calculus and calculus reform have been sources for discussion in post-secondary mathematics. There have been a plethora of suggestions for ways to improve calculus instruction, including a widespread push for the use of graphing technology in the classroom. However, there is yet to be consensus among instructors concerning the topics which should be taught or the methods used to teach them. Nevertheless, the view of having multiple representations of concepts for mathematical learning (graphical, numerical and algebraic) is one that has been growing in significance. Kaput [2, p. 89-103] suggests that the use of multiple representations allows one to emphasize some aspects of more complex concepts while de-emphasizing others thereby helping students make connections and develop a deeper and more flexible understanding of those concepts. If technology is to be effectively utilized in a classroom, students must be able to effectively operate among these various representations. Graphical representations can be easily added to algebraic and numerical representations through the use of the graphing calculators. By linking these representations, we have found that we can enhance the visualization of the topics of

integration of logarithmic functions and inverse trigonometric functions, the focus of this article.

These integration problems are normally tackled in the second semester of a traditional calculus course when integration by parts is introduced. (That is, integration by parts is typically used to find the integrals of logarithmic and inverse trigonometric functions). Boas and Marcus [1, p. 285-286], however, took a different route and found a formulation for integration by parts in terms of inverse functions. Nelsen, used an area model in his *Proof Without Words* [3, p. 287] to illustrate integration by parts. We believe, however, that the integrals of these logarithmic and inverse trigonometric functions can be taught much earlier in the calculus sequence *without* the use of integration by parts.

Many students have relevant knowledge that they do not draw upon in thinking about topics in calculus. Such is the case when investigating the integrals of logarithms and inverse trigonometric functions. We have found that by using the knowledge of inverse functions and the geometric ideas of area, these integrals can be found.

The problems presented in this article are a sampling of those used by the authors in a first semester calculus course. They have been used in courses taught at both the Illinois Mathematics and Science Academy and Kent State University-Stark Campus in a variety of settings and with students of varying abilities. In order to successfully solve these problems, the students must have a solid background in the topics of one-to-one and inverse functions, including logarithmic and inverse trigonometric functions. These topics are typically investigated in prerequisite precalculus courses. Given these prerequisites, the Fundamental Theorem of Calculus is the key to understanding the ideas presented here. It is imperative that the instructor present this theorem, its consequences and its applications carefully and in great detail. The Fundamental Theorem of Calculus has always been a part of any serious calculus course, but often it is simply the mechanical manipulation that is emphasized. By leading students carefully through the proof of this theorem and presenting them with thoughtful problems, we have found that this method does not require any more course time than what should already be devoted to this very important result. The diligent student is rewarded with a sneak preview of the technique of Integration by Parts, but most importantly rewarded with a deeper

understanding of the geometric nature of integration and the omnipresent theme of the Fundamental Theorem of Calculus. After all, the extensions of this theorem range from the concrete higher dimensional analogues of the theorems of Green and Stokes to the abstract and elegant result of Radon and Nikodym.

Typically, problems like problem 1 are given as in-class exercises where the students work together in small groups. The rest, along with their respective hints, are a sample of problems given either as outside exercises, (homework), as part of a take-home test, or as challenging items of an in-class examination. Occasionally, additional hints (maybe a reminder of the fact that area is rotation invariant) are given. When used as outside exercises, students may seek additional help provided that they first explain how they attempted to complete the exercises. In our experience, we have not found these ideas to be beyond the grasp of the conscientious student. Incorporating this material as graded assessments motivates students while imposing no additional time burden to the course. On the contrary its introduction early in the course could result into *saving* time. Calculus students in both of the institutions mentioned above, found the problems "refreshing" and "insightful". One student wrote: "...it was hard at first, but has paid off many times for me...Your classroom style also helped us work extra hard and truly master the material, rather than just learning some formulas like many teachers do."

Our goal is to help students draw upon previously held knowledge and to guide them to perform at increasingly challenging levels as you will see in the problems presented here. As an alternative to straight lecture, we believe interactive learning can provide students with situations that push the boundaries of their abilities and actively engage them in tasks. We accomplish this by breaking problems into workable parts of increasing complexity structured to illustrate the power of the important concepts and techniques mentioned earlier as well as to link algebraic and trigonometric concepts with topics that are not normally handled in this manner.

Integration of logarithmic and inverse trigonometric functions

The Fundamental Theorem of Calculus is a focus here. More often than not, instructors and students alike, downplay the role of the first Fundamental Theorem of Calculus (that is, for continuous f in $[a, b]$ we have $\frac{d}{dx} \int_a^x f(t) dt = f(x)$). Figure 1 illustrates this fact.

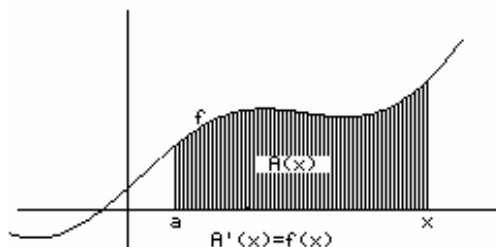


Figure 1: If $A(x)$ is the area under the graph of f from a to x then the derivative of A is f .

The combination of this theorem, together with elementary ideas about integrals and inverse functions is potent. The problems that follow lead one in finding antiderivatives of inverse functions *without* the use of integration by parts.

Problem 1 (Warm up) Suppose that f is continuous and increasing with $f(0) = 1$, $f(2) = 9$ and $\int_0^2 f(r) dr = 8$. Compute $\int_1^9 f^{-1}(s) ds$.

Solution:

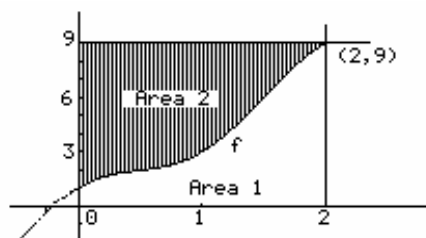


Figure 2: Area 2 is the integral of the inverse function over the interval $[1, 9]$.

After viewing Figure 2, the solution becomes obvious. Notice, $Area\ 1 + Area\ 2$ is the area of the rectangle with vertices $(0,0)$, $(2,0)$, $(2,9)$ and $(0,9)$. Thus, $Area\ 1 + Area\ 2 = 18$.

The integral $\int_1^9 f^{-1}(s) ds$ is simply the shaded area in Figure 2 ($Area\ 2$). Given that

$$Area\ 1 = \int_0^2 f(r) dr = 8, \text{ we conclude } Area\ 2 = \int_1^9 f^{-1}(s) ds = 10.$$

Problem 2 (The antiderivative of the logarithm) *One of the antiderivatives of the natural logarithm is the function F defined by $F(x) = \int_1^x \ln t dt$, for $x > 0$. Find a “nice” formula for $F(x)$ by interpreting it as an area. (Hint: Refer to Problem 1 and DRAW A PICTURE!).*

Solution:

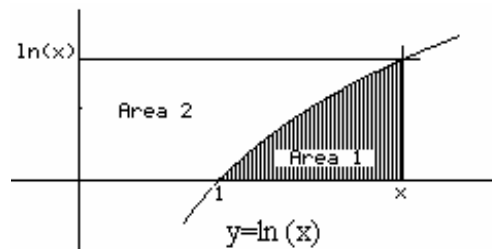


Figure 3: $Area\ 1$ is a geometric representation of an antiderivative for the natural logarithm.

After viewing Figure 3, one can see that $F(x)$ is nothing more than the shaded area of the

figure (i.e. $Area\ 1$). So we have that $F(x) = Area\ 1 = \int_1^x \ln t dt$, while $Area\ 2 = \int_0^{\ln x} e^t dt$.

As with the previous problem the key idea is that areas of rectangles are easy to compute. Indeed the sum of $Area\ 1$ and $Area\ 2$ is the area of the rectangle with vertices $(0,0)$, $(x,0)$, $(x,\ln x)$ and $(0,\ln x)$. So $Area\ 1 + Area\ 2 = x \ln x$. Hence,

$$F(x) = x \ln x - \int_0^{\ln x} e^t dt = x \ln x - e^{\ln x} + 1 = x \ln x - x + 1.$$

Thus we conclude that the general form of an antiderivative of f is

$$\int \ln x dx = x \ln x - x + C$$

where C is an arbitrary constant.

Problem 3 Using the ideas of problem 2 find the following antiderivatives:

a. $\int \tan^{-1} x \, dx$

b. $\int \sin^{-1} x \, dx$

c. $\int \sec^{-1} x \, dx$

Solution:

a. Following the plan of the previous problem, for any real x , we let $F(x) = \int_0^x \tan^{-1} t \, dt$.

Then F is an antiderivative of the inverse tangent function.

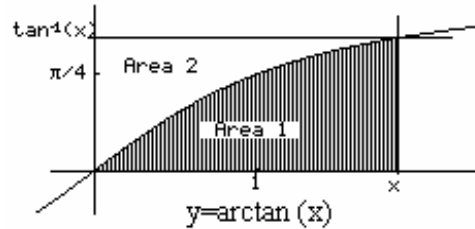


Figure 4: Area 1 is a geometric representation of an antiderivative for the inverse tangent function.

By Figure 4 above, $F(x) = \text{Area 1}$ (i.e. the shaded area).

Furthermore, $\text{Area 1} + \text{Area 2} = x \tan^{-1} x$, with $\text{Area 2} = \int_0^{\tan^{-1} x} \tan t \, dt$. Hence,

$$F(x) = x \tan^{-1} x - \int_0^{\tan^{-1} x} \tan t \, dt = x \tan^{-1} x + \ln |\cos(\tan^{-1} x)| = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2),$$

and so

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.$$

b. Again, for $-1 \leq x \leq 1$, we let $F(x) = \int_0^x \sin^{-1} t \, dt$. Then, F is an antiderivative of the inverse sine function.

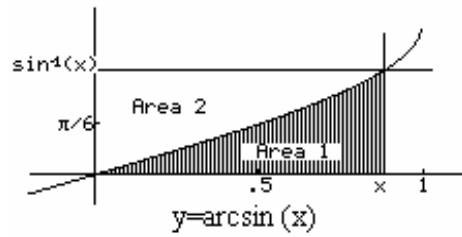


Figure 5: *Area 1* is a geometric representation of an antiderivative for the inverse sine function.

By Figure 5 above, $F(x) = \text{Area 1}$ (i.e. the shaded area). Furthermore, $\text{Area 1} + \text{Area 2} = x \sin^{-1} x$, with $\text{Area 2} = \int_0^{\sin^{-1} x} \sin t \, dt$. Hence,

$$F(x) = x \sin^{-1} x - \int_0^{\sin^{-1} x} \sin t \, dt = x \sin^{-1} x + \cos(\sin^{-1} x) = x \sin^{-1} x + \sqrt{1 - x^2},$$

and so,

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

c. For $x \geq 1$, we let $F(x) = \int_1^x \sec^{-1} t \, dt$.

Then F is an antiderivative of the inverse secant function on the interval $[1, \infty)$.

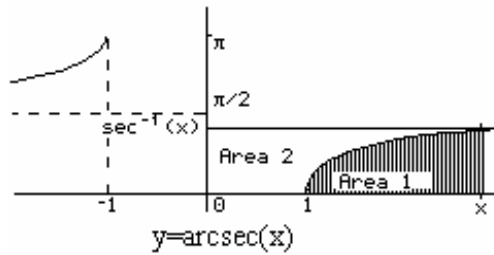


Figure 6: *Area 1* is a geometric representation of an antiderivative for the inverse secant on the interval $[1, \infty)$.

By Figure 6 above, $F(x) = \text{Area 1}$ (i.e. the shaded area).

Furthermore, $\text{Area } 1 + \text{Area } 2 = x \sec^{-1} x$, with $\text{Area } 2 = \int_0^{\sec^{-1} x} \sec t \, dt$. Keeping in mind that for $x \geq 1$, $0 \leq \sec^{-1} x < \pi/2$, and that on the first quadrant the tangent function is non-negative, we have

$$\begin{aligned} F(x) &= x \sec^{-1} x - \int_0^{\sec^{-1} x} \sec t \, dt \\ &= x \sec^{-1} x - \ln \left| \sec(\sec^{-1} x) + \tan(\sec^{-1} x) \right| \\ &= x \sec^{-1} x - \ln \left| x + \sqrt{x^2 - 1} \right| \end{aligned}$$

and so,

$$\int \sec^{-1} x \, dx = x \sec^{-1} x - \ln \left| x + \sqrt{x^2 - 1} \right| + C.$$

For $x \leq -1$, we let $F(x) = \int_{-1}^x \sec^{-1} t \, dt$.

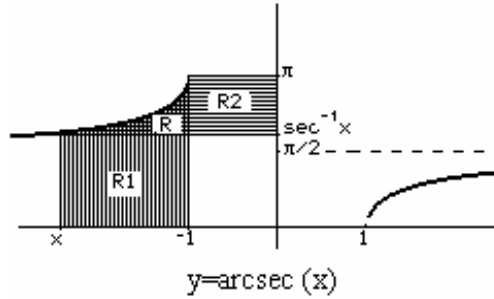


Figure 7: The negative of the sum of the areas of the regions $R1$ and R , represents an antiderivative for the inverse secant on the interval $(-\infty, -1]$.

Then F is an antiderivative of the inverse secant function on the interval $(-\infty, -1]$.

By Figure 7 above, $F(x) = -(R1 + R)$. Furthermore, $R2 + R = \int_{\sec^{-1} x}^{\pi} \sec t \, dt$. Keeping in

mind that for $x \leq -1$, $\frac{\pi}{2} < \sec^{-1} x \leq \pi$, and that on the second quadrant the tangent function is non-positive, we have

$$R2 + R = \int_{\sec^{-1} x}^{\pi} \sec t \, dt = \ln|\sec t + \tan t| \Big|_{t=\sec^{-1} x}^{t=\pi} = -\ln|x - \sqrt{x^2 - 1}|.$$

Hence, $R = -\ln|x - \sqrt{x^2 - 1}| - R2 = -\ln|x - \sqrt{x^2 - 1}| - (\pi - \sec^{-1} x)$. Thus

$$\begin{aligned} F(x) &= -(R1 + R) \\ &= -[(-1 - x)\sec^{-1} x + R] \\ &= (1 + x)\sec^{-1} x + \ln|x - \sqrt{x^2 - 1}| + (\pi - \sec^{-1} x) \\ &= x\sec^{-1} x + \ln|x - \sqrt{x^2 - 1}| + \pi \end{aligned}$$

and so,

$$\begin{aligned} \int \sec^{-1} x \, dx &= x\sec^{-1} x + \ln|x - \sqrt{x^2 - 1}| + C \\ &= x\sec^{-1} x - \ln \frac{1}{|x - \sqrt{x^2 - 1}|} + C \\ &= x\sec^{-1} x - \ln|x + \sqrt{x^2 - 1}| + C \end{aligned}$$

Hence, in both cases

$$\int \sec^{-1} x \, dx = x\sec^{-1} x - \ln|x + \sqrt{x^2 - 1}| + C.$$

Similar arguments can be used to integrate the inverse co-functions (i.e. the inverse cosine, the inverse cosecant and the inverse cotangent). Having students work through these exercises will help to reinforce the concepts in the problems presented above.

In this article, we have attempted to present an alternative method for integrating logarithmic and inverse trigonometric functions. Using the concepts of area and the first Fundamental Theorem of Calculus, students can concretely visualize these integrals. By linking symbolic and geometric representations, these integrals can be found without the use of integration by parts. Thus, they can be introduced earlier than what is traditionally done in a calculus course. Of course, reinforcement of these integrals when the topic of integration by parts is presented will help to strengthen a student's knowledge base.

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BIOGRAPHICAL SKETCH

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